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Solutions of the Gaudin equation and Gaudin algebras

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Abstract

Three well-known solutions of the Gaudin equation are obtained under a set of standard assumptions. By relaxing one of these assumptions, we introduce a class of mutually commuting Hamiltonians based on a different solution of the Gaudin equation. Application of the algebraic Bethe ansatz technique to diagonalize these Hamiltonians reveals a new infinite-dimensional complex Lie algebra.

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1. Introduction

The Bardeen–Cooper–Schrieffer (BCS) pairing model Hamiltonian was diagonalized by Richardson in 1963 in an algebraic way [1]. The step operators introduced in Richardson's solution belong to an infinite-dimensional complex Lie algebra which is usually referred to as the rational Gaudin algebra. It is one of the three algebras which emerged from Gaudin's work during the 1970s [2] who introduced a new class of integrable models starting with the following operators:

$$h_i = \sum_{\substack{j=1 \\ j\neq i}}^{N} \sum_{\alpha=0}^{2} w_{ij}^{\alpha} t_i^{\alpha} t_j^{\alpha}, \tag{1}$$

where w_{ij}^{α} are complex numbers to be determined and t_i^{α} are the generators of N, mutually commuting $SU(2)$ algebras. The latter obey the standard $SU(2)$ commutation relations:

$$\left[t_i^+, t_j^-\right] = 2\delta_{ij} t_j^0, \qquad \left[t_i^0, t_j^{\pm}\right] = \pm\delta_{ij} t_j^{\pm}, \qquad i, j = 1, 2, \ldots, N, \tag{2}$$

where $t_i^{\pm} \equiv t_i^1 \pm i t_i^2$. Gaudin showed that the operators in (1) mutually commute

$$[h_i, h_j] = 0 \tag{3}$$

if and only if the coefficients satisfy the equations:

$$w_{ij}^{\alpha} w_{jk}^{\gamma} + w_{ji}^{\beta} w_{ik}^{\gamma} - w_{ik}^{\alpha} w_{jk}^{\beta} = 0 \tag{4}$$

for all distinct triples (i, j, k) and for all permutations of the upper indices $(0, 1, 2)$. There are three solutions to equation (4) under the following assumptions: (i) The coefficients w_{ij}^{α} are antisymmetric under the exchange of the indices i and j

$$w_{ij}^{\alpha} + w_{ji}^{\alpha} = 0, \tag{5}$$

(ii) each coefficient w_{ij}^{α} can be expressed as a function of the difference between two real parameters u_i and u_j and (iii) each h_i commutes with the operator $T = \sum_j t_j^0$ so that the z component of the total $SU(2)$ is conserved. Gaudin's solutions are given as follows:

$$w_{ij}^{\alpha} = \frac{1}{u_i - u_j} \qquad \text{for} \quad \alpha = 0, 1, 2, \tag{6}$$

$$w_{ij}^{0} = p\cot[p(u_i - u_j)], \qquad w_{ij}^{1} = w_{ij}^{2} = \frac{p}{\sin[p(u_i - u_j)]}, \tag{7}$$

$$w_{ij}^{0} = p\coth[p(u_i - u_j)], \qquad w_{ij}^{1} = w_{ij}^{2} = \frac{p}{\sinh[p(u_i - u_j)]}, \tag{8}$$

which are commonly referred to as rational, trigonometric and hyperbolic solutions, respectively. Here, p is a real parameter[4].

The operator which is obtained by substituting the rational solution into (1) is called rational Gaudin magnet Hamiltonian:

$$h_i^{(r)} = \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{\vec{t}_i \cdot \vec{t}_j}{u_i - u_j}, \tag{9}$$

where

$$\vec{t}_i \cdot \vec{t}_j = t_i^0 t_j^0 + t_i^1 t_j^1 + t_i^2 t_j^2 = t_i^0 t_j^0 + \tfrac{1}{2}\left(t_i^+ t_j^- + t_i^- t_j^+\right).$$

Similarly, substituting the trigonometric and hyperbolic solutions into (1) we obtain

$$h_i^{(t)} = \sum_{\substack{j=1 \\ j\neq i}}^{N} \left(p\cot[p(u_i - u_j)] t_i^0 t_j^0 + \frac{p}{2} \frac{t_i^+ t_j^- + t_i^- t_j^+}{\sin[p(u_i - u_j)]} \right) \tag{10}$$

$$h_i^{(h)} = \sum_{\substack{j=1 \\ j\neq i}}^{N} \left(p\coth[p(u_i - u_j)] t_i^0 t_j^0 + \frac{p}{2} \frac{t_i^+ t_j^- + t_i^- t_j^+}{\sinh[p(u_i - u_j)]} \right) \tag{11}$$

which are called trigonometric and hyperbolic Gaudin magnet Hamiltonians, respectively.

Application of the algebraic Bethe ansatz method to the Gaudin magnets yields the rational, trigonometric and hyperbolic Gaudin algebras [3, 4]. These are infinite-dimensional complex Lie algebras which are related to particular solutions of the classical Yang–Baxter

[4] These solutions are not completely unrelated. Gaudin equation (4) is satisfied for all complex values of p. But only for the real and pure imaginary values of p Hamiltonians in (1) are Hermitian. On the other hand, substituting ip in place of p, one can convert trigonometric and hyperbolic solutions into each other. For this reason, we restrict ourselves to the real values of p. Also note that in the limit $p \to 0$ trigonometric and hyperbolic solutions go to the rational solution.

equation [5]. As a result, each algebra admits a one-parameter family of mutually commuting Hamiltonians $H(\lambda)$. Here λ is a complex parameter which is usually referred to as the spectral parameter. These operators may be identified as the integrals of motion of a quantum system as well as the traces of transfer matrices of a vertex model. Main objective in both cases is to diagonalize them simultaneously and this is achieved by either functional or algebraic Bethe ansatz techniques [6]. Richardson–Gaudin methods have recently found many applications in quantum many-body physics [7].

2. Further solutions of the Gaudin equation

In this paper, we investigate a different solution of the Gaudin equation (4). Under the three assumptions listed in the introduction, all possible solutions are enumerated above. To find new solutions we need to relax one or more of these assumptions. We keep the constraints (ii) and (iii) but generalize the constraint (i) to

$$w_{ij}^{\alpha} + w_{ji}^{\alpha} = -2q, \tag{12}$$

where q is a real parameter[5]. This solution is given by

$$w_{ij}^{\alpha} = q \coth[q(u_i - u_j)] - q \qquad \text{for} \quad \alpha = 0, 1, 2. \tag{13}$$

The operators which are obtained by substituting this new solution into (1) will be denoted by $h_i^{(q)}$:

$$h_i^{(q)} = \sum_{\substack{j=1 \\ j \neq i}}^{N} (q \coth[q(u_i - u_j)] - q) \overrightarrow{t_i} \cdot \overrightarrow{t_j}. \tag{14}$$

Since (13) is a solution of Gaudin equation (4), these operators mutually commute:

$$\left[h_i^{(q)}, h_j^{(q)}\right] = 0. \tag{15}$$

In the limit $q \to 0$, the new solution given by (13) approaches Gaudin's rational solution (6). As a result,

$$\lim_{q \to 0} h_i^{(q)} = h_i^{(r)}. \tag{16}$$

The solution presented here is a different solution of Gaudin equation and the Hamiltonians given by (14) for $q \neq 0$ cannot be obtained from the rational, trigonometric or hyperbolic Gaudin magnet Hamiltonians. To see that, let us assume for a moment that there exists an operator S such that

$$S h_i^{(*)} S^{-1} = h_i^{(q)}, \tag{17}$$

where $h_i^{(*)}$ represents any one of the Hamiltonians given in equations (9)–(11). Suppose we take the sum of both sides of (17):

$$\sum_{i=1}^{N} S h_i^{(*)} S^{-1} = \sum_{i=1}^{N} h_i^{(q)}. \tag{18}$$

Antisymmetry of (5) implies

$$\sum_{i=1}^{N} h_i^{(*)} = 0 \tag{19}$$

[5] In general, (4) is satisfied for all complex values of q. But only for real values of q one obtains Hermitian Hamiltonians in (1).

for rational, trigonometric and hyperbolic magnet Hamiltonians. Hence the sum on the left-hand side is equal to zero. On the other hand, the sum on the right-hand side of (18) is not equal to zero. Instead the new condition given in (12) implies

$$\sum_{i=1}^{N} h_i^{(q)} = -q \sum_{\substack{i,j=1 \\ i\neq j}}^{N} \vec{t}_i \cdot \vec{t}_j, \tag{20}$$

i.e. an operator S satisfying (17) does not exist for $q \neq 0$.

3. Gaudin algebras

If one applies the algebraic Bethe ansatz technique to diagonalize the Gaudin magnet Hamiltonians which correspond to the solutions (6), (7) or (8), one uses the step operators belonging to the rational, trigonometric or hyperbolic Gaudin algebras, respectively. Below we will show that if the algebraic Bethe ansatz method is employed to diagonalize the Hamiltonian of (14), then a different algebra emerges in a completely analogous way. Before introducing this algebra, however, we would like to outline the basic features of the rational Gaudin algebra and its relation to the rational Gaudin magnet Hamiltonian, as an illustration of the technique. Trigonometric and hyperbolic Gaudin algebras and their relation to the trigonometric and hyperbolic Gaudin magnet Hamiltonians follow very similar lines.

The rational Gaudin algebra is generated by three families of operators $J^+(\lambda)$, $J^-(\lambda)$ and $J^0(\lambda)$ parametrized by a complex number λ. Commutators are given by

$$\begin{aligned} [J^+(\lambda), J^-(\mu)] &= 2\frac{J^0(\lambda) - J^0(\mu)}{\lambda - \mu}, \\ [J^0(\lambda), J^\pm(\mu)] &= \pm\frac{J^\pm(\lambda) - J^\pm(\mu)}{\lambda - \mu}, \\ [J^0(\lambda), J^0(\mu)] &= [J^\pm(\lambda), J^\pm(\mu)] = 0. \end{aligned} \tag{21}$$

Commutators of $J^+(\lambda)$, $J^-(\lambda)$ and $J^0(\lambda)$ at the same value of the complex parameter are given by taking the limit $\mu \to \lambda$. From these commutation relations it is easy to show that

$$H(\lambda) = J^0(\lambda)J^0(\lambda) + \tfrac{1}{2}J^+(\lambda)J^-(\lambda) + \tfrac{1}{2}J^-(\lambda)J^+(\lambda) \tag{22}$$

form a one-parameter family of mutually commuting operators:

$$[H(\lambda), H(\mu)] = 0. \tag{23}$$

Starting from a lowest weight vector and using $J^+(\lambda)$ as step operators, one can diagonalize these operators simultaneously. Lowest weight vector $|0\rangle$ by definition satisfies

$$J^-(\lambda)|0\rangle = 0, \qquad \text{and} \qquad J^0(\lambda)|0\rangle = W(\lambda)|0\rangle \tag{24}$$

for every λ. Here $W(\lambda)$ is a complex-valued function. The state $|0\rangle$ itself is an eigenvector of $H(\lambda)$:

$$H(\lambda)|0\rangle = E_0(\lambda)|0\rangle \tag{25}$$

with the eigenvalue

$$E_0(\lambda) = W(\lambda)^2 - W'(\lambda), \tag{26}$$

where the prime denotes derivative with respect to λ. In addition

$$|\xi_1, \xi_2, \ldots, \xi_n\rangle \equiv J^+(\xi_1)J^+(\xi_2)\cdots J^+(\xi_n)|0\rangle \tag{27}$$

is an eigenvector of $H(\lambda)$ if the quantities $\xi_1, \xi_2, \ldots, \xi_n$ satisfy the set of Bethe ansatz equations

$$W(\xi_\alpha) = \sum_{\substack{\beta=1 \\ (\beta\neq\alpha)}}^{n} \frac{1}{\xi_\alpha - \xi_\beta} \qquad \text{for} \quad \alpha = 1, 2, \ldots, n. \tag{28}$$

(For an analytic solution of these equations see e.g. [8] and [9].) The corresponding eigenvalues will be

$$E_n(\lambda) = E_0(\lambda) - 2\sum_{\alpha=1}^{n} \frac{W(\lambda) - W(\xi_\alpha)}{\lambda - \xi_\alpha}. \tag{29}$$

There exists a realization of the rational Gaudin algebra in terms of the $SU(2)$ generators of (2), given by

$$J^0(\lambda) = \sum_{i=1}^{N} \frac{t_i^0}{u_i - \lambda} \qquad \text{and} \qquad J^\pm(\lambda) = \sum_{i=1}^{N} \frac{t_i^\pm}{u_i - \lambda}. \tag{30}$$

Here $u_1, u_2, \ldots, u_N$ are arbitrary real numbers which are all different from each other and N is a nonnegative integer[6]. Assuming that the eigenvalues of the Casimir operator of the jth $SU(2)$ are $s_j(s_j+1)$, the corresponding $W(\lambda)$ is given by

$$W(\lambda) = \sum_{i=1}^{N} \frac{-s_i}{u_i - \lambda}. \tag{31}$$

In this realization $H(\lambda)$ is given by

$$H(\lambda) = \sum_{i,j=1}^{N} \frac{\overrightarrow{t_i} \cdot \overrightarrow{t_j}}{(u_i - \lambda)(u_j - \lambda)}. \tag{32}$$

We see that $H(\lambda)$ has simple poles on the real axis. Residues of $H(\lambda)$ at the points $\lambda = u_i$ are proportional to the rational Gaudin magnet Hamiltonians given in (9):

$$-\tfrac{1}{2}\,\mathrm{Res}\{H(\lambda))\}_{\lambda=u_i} = h_i^{(r)}. \tag{33}$$

Equation (23) implies

$$\left[H(\lambda), h_i^{(r)}\right] = 0, \qquad \text{and} \qquad \left[h_i^{(r)}, h_j^{(r)}\right] = 0. \tag{34}$$

As a result, rational Gaudin magnet Hamiltonians $h_i^{(r)}$ are automatically diagonalized together with $H(\lambda)$. It is easy to read off the eigenvalues of the rational Gaudin magnet Hamiltonians from the eigenvalues of $H(\lambda)$ as follows:

$$h_i^{(r)}|0\rangle = E_{i,0}^{(r)}|0\rangle, \tag{35}$$

where

$$E_{i,0}^{(r)} = \sum_{\substack{j=1 \\ (j\neq i)}}^{N} \frac{s_i s_j}{u_i - u_j} \tag{36}$$

and

$$h_i^{(r)}|\xi_1, \xi_2, \ldots, \xi_n\rangle = E_{i,n}^{(r)}|\xi_1, \xi_2, \ldots, \xi_n\rangle, \tag{37}$$

[6] In general $u_1, u_2, \ldots, u_N$ can be complex and they need not be different from each other. But in most physical applications we are interested in those realizations for which $u_1, u_2, \ldots, u_N$ are real and different from each other. Reality guarantees that $J^+(\lambda)^\dagger = J^-(\lambda^*)$ and $J^0(\lambda)^\dagger = J^0(\lambda^*)$.

where

$$E_{i,n}^{(r)} = E_{i,0}^{(r)} - s_i \sum_{\alpha=1}^{n} \frac{1}{u_i - \xi_\alpha}. \tag{38}$$

Here $|0\rangle$ is the lowest weight state and ξ_α are the solutions of the Bethe ansatz equation (28).

We next search for an operator $H^{(q)}(\lambda)$, the residue of which gives us the operators $h_i^{(q)}$ (cf equation (33)). We wish to write such an operator using an algebra similar to that in (21). Below we show that such an algebra exists and its generators $J_q^0(\lambda)$, $J_q^+(\lambda)$ and $J_q^-(\lambda)$ satisfy the commutation relations given as

$$\begin{aligned}
\left[J_q^+(\lambda), J_q^-(\mu)\right] &= 2q\frac{J_q^0(\lambda) - J_q^0(\mu)}{\tanh[q(\lambda-\mu)]} + 2q\left(J_q^0(\lambda) + J_q^0(\mu)\right),\\
\left[J_q^0(\lambda), J_q^\pm(\mu)\right] &= \pm q\frac{J_q^\pm(\lambda) - J_q^\pm(\mu)}{\tanh[q(\lambda-\mu)]} \pm q\left(J_q^\pm(\lambda) + J_q^\pm(\mu)\right),\\
\left[J_q^0(\lambda), J_q^0(\mu)\right] &= \left[J_q^\pm(\lambda), J_q^\pm(\mu)\right] = 0.
\end{aligned} \tag{39}$$

We observe that, in the limit $q \to 0$, commutators in (39) approach the commutators of Gaudin algebra given by (21). It should nevertheless be emphasized that the algebra of (39) is not a q-deformed algebra, but an ordinary Lie algebra. We can show that

$$H^{(q)}(\lambda) = J_q^0(\lambda)J_q^0(\lambda) + \tfrac{1}{2}J_q^+(\lambda)J_q^-(\lambda) + \tfrac{1}{2}J_q^-(\lambda)J_q^+(\lambda) \tag{40}$$

form a one-parameter commutative family:

$$[H^{(q)}(\lambda), H^{(q)}(\mu)] = 0. \tag{41}$$

We can simultaneously diagonalize them starting from a lowest weight vector satisfying

$$J_q^-(\lambda)|0\rangle = 0, \qquad J_q^0(\lambda)|0\rangle = W_q(\lambda)|0\rangle, \tag{42}$$

where $W_q(\lambda)$ is a complex-valued function. It can be shown that $|0\rangle$ is an eigenvector of $H^{(q)}(\lambda)$ with the eigenvalue

$$E_0^{(q)}(\lambda) = W_q(\lambda)^2 - W_q'(\lambda) - 2qW_q(\lambda). \tag{43}$$

In general, one can write a Bethe ansatz form for the eigenvectors of $H^{(q)}(\lambda)$ as

$$|\xi_1, \xi_2, \ldots, \xi_n\rangle \equiv J_q^+(\xi_1)J_q^+(\xi_2)\cdots J_q^+(\xi_n)|0\rangle. \tag{44}$$

For this to be an eigenvector of $H^{(q)}(\lambda)$, the complex numbers $\xi_1, \xi_2, \ldots, \xi_n$ must satisfy the following system of equations:

$$W_q(\xi_\alpha) = \sum_{\substack{\beta=1\\(\beta\neq\alpha)}}^{n} q(\coth[q(\xi_\alpha - \xi_\beta)] - 1) \qquad \text{for} \quad \alpha = 1, 2, \ldots, n. \tag{45}$$

In this case, eigenvalue associated with the state $|\xi_1, \xi_2, \ldots, \xi_n\rangle$ is given by

$$E_n^{(q)}(\lambda) = E_0^{(q)}(\lambda) - 2\sum_{\alpha=1}^{n} q(\coth[q(\lambda - \xi_\alpha)] - 1)(W_q(\lambda) - W_q(\xi_\alpha)). \tag{46}$$

This algebra admits two realizations in terms of the $SU(2)$ generators of (2). These are given by

$$J_q^{\pm,0}(\lambda) = \sum_{i=1}^{N} q(\coth[q(u_i - \lambda)] + 1)t_i^{\pm,0} \tag{47}$$

and

$$J_q^{\pm,0}(\lambda) = \sum_{i=1}^{N} q(\tanh[q(u_i - \lambda)] + 1) t_i^{\pm,0}. \tag{48}$$

In the limit $q \to 0$, the first realization goes to the realization of the rational Gaudin algebra given by (30) whereas the second realization vanishes.

In the first realization, equation (47), the operator $H^{(q)}(\lambda)$ becomes

$$H^{(q)}(\lambda) = \sum_{i,j=1}^{N} (q\coth[q(u_i - \lambda)] + q)(q\coth[q(u_j - \lambda)] + q)\overrightarrow{t_i} \cdot \overrightarrow{t_j}. \tag{49}$$

We see that $H^{(q)}(\lambda)$ has simple poles on real axis at the points $\lambda = u_i$. It is easy to show that $-1/2$ times the residue of $H^{(q)}(\lambda)$ at the point $\lambda = u_i$ is

$$-\tfrac{1}{2}\,\mathrm{Res}\{H_q(\lambda)\}_{\lambda=u_i} = h_i^{(q)} - q\overrightarrow{t_i} \cdot \overrightarrow{t_i}. \tag{50}$$

Since $\overrightarrow{t_i} \cdot \overrightarrow{t_i}$ commutes with every $h_j^{(q)}$ and $H^{(q)}(\lambda)$, (41) implies

$$\left[H^{(q)}(\lambda), h_i^{(q)}\right] = 0, \qquad \text{and} \qquad \left[h_i^{(q)}, h_j^{(q)}\right] = 0. \tag{51}$$

As a result, the operators $h_i^{(q)}$ are automatically diagonalized together with $H^{(q)}(\lambda)$. As in the case of the rational Gaudin algebra, one can read off the eigenvalues of $h_i^{(q)}$ from the eigenvalues of $H^{(q)}(\lambda)$:

$$h_i^{(q)}|0\rangle = E_{0,i}^{(q)}|0\rangle, \tag{52}$$

where

$$E_{0,i}^{(q)} = \sum_{\substack{j=1\\(j\neq i)}}^{N} s_i s_j q(\coth[q(u_i - u_j)] - 1) \tag{53}$$

and

$$h_i^{(q)}|\xi_1, \xi_2, \ldots, \xi_n\rangle = E_{n,i}^{(q)}|\xi_1, \xi_2, \ldots, \xi_n\rangle, \tag{54}$$

where

$$E_{n,i}^{(q)} = E_{0,i}^{(q)} - s_i \sum_{\alpha=1}^{n} q(\coth[q(u_i - \xi_\alpha)] - 1). \tag{55}$$

Details of the derivation of these eigenvalues are given in the appendix.

4. Linear *r*-matrix structure

The rational, trigonometric and hyperbolic Gaudin algebras have a linear *r*-matrix structure. This means that one can write the commutators of these algebras in the following matrix form:

$$[L(\lambda) \otimes I, I \otimes L(\mu)] + [r(\lambda - \mu), L(\lambda) \otimes I + I \otimes L(\mu)] = 0, \tag{56}$$

where I is the 2×2 identity matrix, $r(\lambda - \mu)$ is the r-matrix described below and $L(\lambda)$ is given by

$$L(\lambda) = \begin{pmatrix} J^0(\lambda) & J^+(\lambda) \\ J^+(\lambda) & -J^0(\lambda) \end{pmatrix}. \tag{57}$$

$J^0(\lambda)$, $J^+(\lambda)$ and $J^-(\lambda)$ are generators of the rational, trigonometric or hyperbolic Gaudin algebras. In (56), the term $[L(\lambda) \otimes I, I \otimes L(\mu)]$ is equal to a 4×4 matrix whose elements are various commutators between the generators $J^0(\lambda)$, $J^+(\lambda)$ and $J^-(\lambda)$. On the other hand $r(\lambda - \mu)$ is a $Mat_2(\mathbb{C}) \otimes Mat_2(\mathbb{C})$ matrix-valued function of $\lambda - \mu$ which carries the information about the structure constants of the algebra. For instance, for the rational Gaudin algebra, the r-matrix is given by

$$r(\lambda - \mu) = \frac{1}{\lambda - \mu} P, \tag{58}$$

where P is the permutation matrix on $\mathbb{C}^2 \otimes \mathbb{C}^2$, which is equal to

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{59}$$

It is customary to introduce the following three mappings:

$$\varphi^{12} : a \otimes b \longrightarrow a \otimes b \otimes I, \tag{60}$$

$$\varphi^{13} : a \otimes b \longrightarrow a \otimes I \otimes b, \tag{61}$$

$$\varphi^{23} : a \otimes b \longrightarrow I \otimes a \otimes b, \tag{62}$$

where a, b are 2×2 matrices and I is the 2×2 identity matrix. Then r^{ij} are defined as follows:

$$r^{12} = \varphi^{12}[r(\lambda - \mu)], \qquad r^{13} = \varphi^{13}[r(\lambda - \sigma)], \qquad r^{23} = \varphi^{23}[r(\mu - \sigma)]. \tag{63}$$

Since r-matrix carries information about the structure constants of the algebra, Jacobi identity leads to the equality

$$[r^{13}, r^{23}] + [r^{12}, r^{13}] + [r^{12}, r^{23}] = 0 \tag{64}$$

which is the well-known classical Yang–Baxter equation (for solutions of this equation see [10], for a review see [11, 12]). Equation (64) guarantees the mutual commutativity of the operators $H(\lambda)$ which can now be written as

$$H(\lambda) = \tfrac{1}{2}\,\mathrm{Tr}\{L(\lambda)^2\}. \tag{65}$$

Trigonometric and hyperbolic Gaudin algebras also have the r-matrix structure described above and their r-matrices are also solutions of the classical Yang–Baxter equation.

In order to see if one can repeat the same procedure for the algebra we introduced in (39), we study a matrix of the form

$$L_q(\lambda) = \begin{pmatrix} J_q^0(\lambda) & J_q^+(\lambda) \\ J_q^+(\lambda) & -J_q^0(\lambda) \end{pmatrix}. \tag{66}$$

We then substitute it into the equation

$$[L_q(\lambda) \otimes I, I \otimes L_q(\mu)] + [r(\lambda - \mu), L_q(\lambda) \otimes I + I \otimes L_q(\mu)] = 0, \tag{67}$$

assuming the existence of such an r-matrix. One can compute the left-hand side of this equation and then set all the components equal to zero to find the components r^{ij} of this r-matrix. For instance, the (21) and (13) elements are equal to

$$(r_{22} - r_{11} + q\coth[q(\lambda - \mu)] - q) J_q^-(\mu) + (r_{23} - q\coth[q(\lambda - \mu)] - q) J_q^-(\lambda)$$
$$- r_{41} J_q^+(\lambda) + 2 r_{21} J_q^0(\mu)$$

and

$$(r_{11} - r_{33} - q\coth[q(\lambda-\mu)] - q)J_q^+(\lambda) - (r_{23} - q\coth[q(\lambda-\mu)] + q)J_q^+(\mu)$$
$$+ r_{14}J_q^-(\mu) - 2r_{13}J_q^0(\lambda),$$

respectively. Setting these elements equal to zero requires setting the coefficients of all the algebra elements equal to zero. The coefficient of $J_q^-(\lambda)$ in the first equation and the coefficient of $J_q^+(\mu)$ in the second equation give

$$r_{23} - q\coth[q(\lambda-\mu)] - q = 0 \qquad \text{and} \qquad r_{23} - q\coth[q(\lambda-\mu)] + q = 0, \tag{68}$$

respectively. But these equations are incompatible unless $q = 0$. Repeating the same calculations for the other components, one can conclude that there is no r-matrix which satisfies (67). Instead of the $L_q(\lambda)$ matrix in (66), one may try to find a more general form of $L_q(\lambda)$ matrix for which an r-matrix can be found to satisfy (67). But we were unable to find such an $L_q(\lambda)$ matrix[7]. On the other hand we were able to show that one can write the commutators of the algebra introduced in (39) in the following form:

$$[L_q(\lambda)\otimes I, I\otimes L_q(\mu)] + [r_q(\lambda-\mu), L_q(\lambda)\otimes I] + [r_{-q}(\lambda-\mu), I\otimes L_q(\mu)] = 0. \tag{69}$$

Here $L_q(\lambda)$ is given by (66) and the r_q-matrix is given by

$$r_q(\lambda-\mu) = (q\coth[q(\lambda-\mu)] + q)P. \tag{70}$$

P is the permutation matrix introduced in (59). As in the case of the r-matrices of rational, trigonometric and hyperbolic Gaudin algebras, the r_q-matrix carries information about the structure constants of the algebra of (39). r_q^{ij} matrices, defined by similar maps as in equations (60)–(63), satisfy the following equation:

$$\left[r_{-q}^{13}, r_q^{23}\right] + \left[r_{-q}^{12}, r_q^{23}\right] + \left[r_{-q}^{12}, r_q^{13}\right] = 0. \tag{71}$$

We see that the r-matrix structure of the algebra introduced in (39) is not the same as the r-matrix structures of the rational, trigonometric and hyperbolic Gaudin algebras. Nevertheless, it shares with the other three algebras, the crucial property of admitting a one-parameter family of mutually commuting operators $H^{(q)}(\lambda)$. Similar to (65), one can write this operator as the trace of the square of $L_q(\lambda)$:

$$H^{(q)}(\lambda) = \tfrac{1}{2}\,\mathrm{Tr}\{L_q(\lambda)^2\}. \tag{72}$$

Note that in the limit $q \to 0$, the algebra of (39) approaches the rational Gaudin algebra. In this limit both r_q and r_{-q} go to the r-matrix of the rational Gaudin algebra given in (58) and from (71) we recover the classical Yang–Baxter equation (64).

5. Conclusion

In this paper, by relaxing one of the conditions imposed by Gaudin, we presented a different solution to the Gaudin equation and wrote the corresponding set of mutually commuting Hamiltonians. We also identified the related infinite-dimensional Lie algebra. This algebra allows a one-parameter family of mutually commuting Hamiltonians $H^{(q)}(\lambda)$ parametrized by a complex spectral parameter. We diagonalized these Hamiltonians using the algebraic Bethe

[7] For instance, for a matrix of the form

$$L_q(\lambda) = \begin{pmatrix} \alpha J^0(\lambda) & \beta J^+(\lambda) + \gamma J^-\lambda) \\ \beta J^-(\lambda) - \gamma J^+(\lambda) & -\alpha J^0(\lambda) \end{pmatrix},$$

$\mathrm{Tr}[L_q(\lambda)L_q^\dagger(\lambda)]$ is proportional to $H^{(q)}(\lambda)$ when $\alpha^2 = \beta^2 + \gamma^2$. However, one can similarly show that there is no solution to (67) for this $L_q(\lambda)$ either.

ansatz method and constructed the eigenvectors and eigenvalues. The procedure is parallel to the other known Gaudin algebras.

A different generalization of the Gaudin algebras was recently introduced in [13] that is based on the rational, trigonometric, and hyperbolic solutions of the Gaudin equations. Our generalization is based on a different solution of the Gaudin equation obtained under different assumptions.

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Appendix. Computation of the eigenvalues

To see that the lowest weight state given by (42) is an eigenstate of $H^{(q)}(\lambda)$ we first use the commutators (39) to write $H^{(q)}(\lambda)$ in the following form:

$$\begin{aligned} H^{(q)}(\lambda) &= J^0(\lambda)J^0(\lambda) + J^+(\lambda)J^-(\lambda) + \frac{1}{2}[J^-(\lambda), J^+(\lambda)] \\ &= J^0(\lambda)J^0(\lambda) + J^+(\lambda)J^-(\lambda) - \frac{1}{2}\lim_{\mu\to\lambda} 2q\left(\frac{J^0(\lambda)-J^0(\mu)}{\tanh[q(\lambda-\mu)]} + J^0(\lambda) + J^0(\mu)\right). \end{aligned} \tag{A.1}$$

Then the action of $H^{(q)}(\lambda)$ on the lowest weight state is

$$H^{(q)}(\lambda)|0\rangle = (W(\lambda)^2 - W'(\lambda) - 2qW(\lambda))|0\rangle. \tag{A.2}$$

We see that the lowest weight state is an eigenstate of $H^{(q)}(\lambda)$ with the eigenvalue $E_0(\lambda)$ given by (43).

Now consider the state $|\xi\rangle = J^+(\xi)|0\rangle$. Action of $H^{(q)}(\lambda)$ on $J^+(\xi)|0\rangle$ is

$$\begin{aligned} H^{(q)}(\lambda)J^+(\xi)|0\rangle &= 2q\,(\coth[q(\lambda-\xi)] - 1)\,W(\xi)J^+(\lambda)|0\rangle \\ &\quad + (E_0(\lambda) - 2qW(\lambda)\coth[q(\lambda-\xi)] + 2qW(\lambda))J^+(\xi)|0\rangle. \end{aligned} \tag{A.3}$$

We see that $H^{(q)}(\lambda)J^+(\xi)|0\rangle$ is a superposition of the states $J^+(\lambda)|0\rangle$ and $J^+(\xi)|0\rangle$. Therefore, for a generic ξ, the vector $J^+(\xi)|0\rangle$ is not an eigenstate of $H^{(q)}(\lambda)$. But if ξ is a root of the $W(\lambda)$ then the coefficient of $J^+(\lambda)|0\rangle$ vanishes. In this case $J^+(\xi)|0\rangle$ is an eigenstate of $H^{(q)}(\lambda)$ with the eigenvalue $E_1(\lambda)$ given by

$$E_1^{(q)}(\lambda) = E_0(\lambda) - 2\,(q\coth[q(\lambda-\xi)] - q)\,W(\lambda). \tag{A.4}$$

This is the energy one finds by substituting $n = 1$ and $W(\xi) = 0$ in equation (46).

For $n > 1$, action of $H^{(q)}(\lambda)$ on the state $J^+(\xi_1)J^+(\xi_2)\cdots J^+(\xi_n)|0\rangle$ is given by

$$\begin{aligned} H^{(q)}(\lambda)J^+(\xi_1)J^+(\xi_2)\cdots J^+(\xi_n)|0\rangle &= J^+(\xi_1)J^+(\xi_2)\cdots J^+(\xi_n)H^{(q)}(\lambda)|0\rangle \\ &\quad - \sum_{\alpha=1}^{n} 2q(\coth[q(\lambda-\xi_\alpha)] - 1)\left(\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{n} q(\coth[q(\xi_\alpha-\xi_\beta)] - 1) - W(\xi_\alpha)\right) \\ &\quad \times J^+(\xi_1)\cdots J^+(\xi_{\alpha-1})J^+(\lambda)J^+(\xi_{\alpha+1})\cdots J^+(\xi_n)|0\rangle \end{aligned}$$

$$+\sum_{\alpha=1}^{n} 2q\,(\coth[q(\lambda-\xi_\alpha)]-1)\left(\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{n} q(\coth[q(\xi_\alpha-\xi_\beta)]-1)-W(\lambda)\right)$$
$$\times J^+(\xi_1)J^+(\xi_2)\cdots J^+(\xi_n)|0\rangle. \tag{A.5}$$

Here we see that $H^{(q)}(\lambda)J^+(\xi_1)J^+(\xi_2)\cdots J^+(\xi_n)|0\rangle$ is a superposition of the states

$$J^+(\xi_1)\cdots J^+(\xi_{\alpha-1})J^+(\lambda)J^+(\xi_{\alpha+1})\cdots J^+(\xi_n)|0\rangle \tag{A.6}$$

for $\alpha=1,2,\ldots,n$ and the state

$$J^+(\xi_1)J^+(\xi_2)\cdots J^+(\xi_n)|0\rangle. \tag{A.7}$$

Coefficients of the states $J^+(\xi_1)\cdots J^+(\lambda)\cdots J^+(\xi_n)|0\rangle$ vanish when the conditions (45) are satisfied. Consequently, $J^+(\xi_1)J^+(\xi_2)\cdots J^+(\xi_n)|0\rangle$ become an eigenstate of $H^{(q)}(\lambda)$ and its eigenvalue is given by (46).

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